The number of chains of subgroups of a finite elementary abelian p-group

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Abstract

In this short note we give a formula for the number of chains of subgroups of a finite elementary abelian p-group. This completes our previous work [5].

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Key words: chains of subgroups, fuzzy subgroups, finite elementary abelian p-groups, recurrence relations.

1 Introduction

Let G be a group. A *chain of subgroups* of G is a set of subgroups of G totally ordered by set inclusion. A chain of subgroups of G is called *rooted* (more exactly G*-rooted*) if it contains G. Otherwise, it is called *unrooted*. Notice that there is a bijection between the set of G*-rooted* chains of subgroups of G and the set of distinct fuzzy subgroups of G (see e.g. [5]), which is used to solve many computational problems in fuzzy group theory.

The starting point for our discussion is given by the paper [5], where a formula for the number of rooted chains of subgroups of a finite cyclic group is obtained. This leads in [3] to precise expression of the well-known central Delannoy numbers in an arbitrary dimension and has been simplified in [2]. Some steps in order to determine the number of rooted chains of subgroups of a finite elementary abelian p-group are also made in [5]. Moreover, this counting problem has been naturally extended to non-abelian groups in other

works, such as [1, 4]. The purpose of the current note is to improve the results of [5], by indicating an explicit formula for the number of rooted chains of subgroups of a finite elementary abelian p-group.

Given a finite group G, we will denote by $\mathcal{C}(G)$, $\mathcal{D}(G)$ and $\mathcal{F}(G)$ the collection of all chains of subgroups of G, of unrooted chains of subgroups of G and of G-rooted chains of subgroups of G, respectively. Put $C(G) = |\mathcal{C}(G)|$, $D(G) = |\mathcal{D}(G)|$ and $F(G) = |\mathcal{F}(G)|$. The connections between these numbers have been established in [2], namely:

Theorem 1. *Let G be a finite group. Then*

$$F(G) = D(G) + 1 \textit{ and } C(G) = F(G) + D(G) = 2F(G) - 1\,.$$

In the following let p be a prime, n be a positive integer and $\mathbb{Z}_p^n$ be an elementary abelian p-group of rank n (that is, a direct product of n copies of $\mathbb{Z}_p$). First of all, we recall a well-known group theoretical result that gives the number $a_{n,p}(k)$ of subgroups of order p^k in $\mathbb{Z}_p^n$, $k = 0, 1, ..., n$.

Theorem 2. *For every $k = 0, 1, ..., n$, we have*

$$a_{n,p}(k) = \frac{(p^n-1)\cdots(p-1)}{(p^k-1)\cdots(p-1)(p^{n-k}-1)\cdots(p-1)}\,.$$

Our main result is the following.

Theorem 3. *The number of rooted chains of subgroups of the elementary abelian p-group $\mathbb{Z}_p^n$ is*

$$F(\mathbb{Z}_p^n) = 2+2f(n)\sum_{k=1}^{n-1}\;\sum_{1\le i_1<i_2<\ldots<i_k\le n-1}\frac{1}{f(n-i_k)f(i_k-i_{k-1})\cdots f(i_2-i_1)f(i_1)}\,,$$

where $f : \mathbb{N} \longrightarrow \mathbb{N}$ is the function defined by $f(0) = 1$ and $f(r) = \prod_{s=1}^{r}(p^s-1)$ for all $r \in \mathbb{N}^*$.

Obviously, explicit formulas for $C(\mathbb{Z}_p^n)$ and $D(\mathbb{Z}_p^n)$ also follow from Theorems 1 and 2. By using a computer algebra program, we are now able to calculate the first terms of the chain $f_n = F(\mathbb{Z}_p^n)$, $n \in \mathbb{N}$, namely:

- $f_0 = 1$;

- $f_1 = 2$;

- $f_2 = 2p + 4$;

- $f_3 = 2p^3 + 8p^2 + 8p + 8$;

- $f_4 = 2p^6 + 12p^5 + 24p^4 + 36p^3 + 36p^2 + 24p + 16$.

Finally, we remark that the above f_3 is in fact the number $a_{3,p}$ obtained by a direct computation in Corollary 10 of [5].

2 Proof of Theorem 3

We observe first that every rooted chain of subgroups of $\mathbb{Z}_p^n$ are of one of the following types:

$$G_1 \subset G_2 \subset ... \subset G_m = \mathbb{Z}_p^n \text{ with } G_1 \neq 1 \tag{1}$$

and

$$1 \subset G_2 \subset ... \subset G_m = \mathbb{Z}_p^n \,. \tag{2}$$

It is clear that the numbers of chains of types (1) and (2) are equal. So

$$f_n = 2x_n \,, \tag{3}$$

where x_n denotes the number of chains of type (2). On the other hand, such a chain is obtained by adding $\mathbb{Z}_p^n$ to the chain

$$1 \subset G_2 \subset ... \subset G_{m-1},$$

where G_{m-1} runs over all subgroups of $\mathbb{Z}_p^n$. Moreover, G_{m-1} is also an elementary abelian p-group, say $G_{m-1} \cong \mathbb{Z}_p^k$ with $0 \leq k \leq n$. These show that the chain x_n, $n \in \mathbb{N}$, satisfies the following recurrence relation

$$x_n = \sum_{k=0}^{n-1} a_{n,p}(k) x_k \,, \tag{4}$$

which is more facile than the recurrence relation founded by applying the Inclusion-Exclusion Principle in Theorem 9 of [5].

Next we prove that the solution of (4) is given by

$$x_n = 1+\sum_{k=1}^{n-1} \sum_{1\le i_1<i_2<\ldots<i_k\le n-1} a_{n,p}(i_k)a_{i_k,p}(i_{k-1})\cdots a_{i_2,p}(i_1)\,. \tag{5}$$

We will proceed by induction on n. Clearly, (5) is trivial for $n=1$. Assume that it holds for all $k<n$. One obtains

$$x_n = \sum_{k=0}^{n-1} a_{n,p}(k)x_k = 1+\sum_{k=1}^{n-1} a_{n,p}(k)x_k =$$

$$= 1+\sum_{k=1}^{n-1} a_{n,p}(k)\left(1+\sum_{r=1}^{k-1}\sum_{1\le i_1<i_2<\ldots<i_r\le k-1} a_{k,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1)\right) =$$

$$= 1+\sum_{k=1}^{n-1} a_{n,p}(k)+\sum_{k=1}^{n-1} a_{n,p}(k)\sum_{r=1}^{k-1}\sum_{1\le i_1<i_2<\ldots<i_r\le k-1} a_{k,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1) =$$

$$= 1+\sum_{k=1}^{n-1} a_{n,p}(k)+\sum_{k=1}^{n-1} a_{n,p}(k)\sum_{r=1}^{n-2}\sum_{1\le i_1<i_2<\ldots<i_r\le k-1} a_{k,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1) =$$

$$= 1+\sum_{k=1}^{n-1} a_{n,p}(k)+\sum_{k=1}^{n-1} a_{n,p}(k)\sum_{r=2}^{n-1}\sum_{1\le i_1<i_2<\ldots<i_{r-1}\le k-1} a_{k,p}(i_{r-1})a_{i_{r-1},p}(i_{r-2})\cdots a_{i_2,p}(i_1) =$$

$$= 1+\sum_{1\le i_1\le n-1} a_{n,p}(i_1)+\sum_{r=2}^{n-1}\sum_{k=1}^{n-1} a_{n,p}(k)\sum_{1\le i_1<i_2<\ldots<i_{r-1}\le k-1} a_{k,p}(i_{r-1})a_{i_{r-1},p}(i_{r-2})\cdots a_{i_2,p}(i_1) =$$

$$= 1+\sum_{1\le i_1\le n-1} a_{n,p}(i_1)+\sum_{r=2}^{n-1}\sum_{1\le i_1<i_2<\ldots<i_r\le n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1) =$$

$$= 1+\sum_{r=1}^{n-1}\sum_{1\le i_1<i_2<\ldots<i_r\le n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1)\,,$$

as desired.

Since by Theorem 2

$$a_{n,p}(k)=\frac{(p^n-1)\cdots(p-1)}{(p^k-1)\cdots(p-1)(p^{n-k}-1)\cdots(p-1)}=\frac{f(n)}{f(k)f(n-k)}\,,\forall\, 0\le k\le n\,,$$

the equalities (3) and (5) imply that

$$f_n=2+2f(n)\sum_{k=1}^{n-1}\sum_{1\le i_1<i_2<\ldots<i_k\le n-1}\frac{1}{f(n-i_k)f(i_k-i_{k-1})\cdots f(i_2-i_1)f(i_1)}\,,$$

completing the proof. □

References

[1] Davvaz, B., Ardekani, R.K., *Counting fuzzy subgroups of a special class of non-abelian groups of order* p^3, Ars Combin. **103** (2012), 175-179.

[2] Oh, J.M., *The number of chains of subgroups of a finite cyclic group*, European J. Combin. **33** (2012), 259-266.

[3] Tărnăuceanu, M., *The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers*, European J. Combin. **30** (2009), 283-287, doi: 10.1016/j.ejc.2007.12.005.

[4] Tărnăuceanu, M., *Classifying fuzzy subgroups of finite nonabelian groups*, Iran. J. Fuzzy Syst. **9** (2012), 33-43.

[5] Tărnăuceanu, M., Bentea, L., *On the number of fuzzy subgroups of finite abelian groups*, Fuzzy Sets and Systems **159** (2008), 1084-1096, doi: 10.1016/j.fss.2007.11.014.

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